Derivative of a Function

-Before we defined the slope of a curve $y = f\left(x\right)$ at a point where $x = a$ to be

$$m = \lim_{h \to 0} \frac{f\left(a+h\right) - f\left(a\right)}{h}$$

-Where it exists, this limit is called the derivative of f at a.

Derivative

-The derivative of the function f with respect to the variable x is the function f' whose value at x is

$$f'\left(x\right) = \lim_{h \to 0} \frac{f\left(x+h\right) - f\left(x\right)}{h}$$

provided the limit exists.

-The domain of f', the set of points in the domain of f' for which the limit exists, may be smaller than the domain of f.

-If $f'\left(x\right)$ exists, we say that f has a derivative (is differentiable) at x.

-A function that is differentiable at every point in its domain is a differentiable function.

Apply the Definition

Differentiate $f\left(x\right) = x^3$

$$f'\left(x\right) = \lim_{h \to 0} \frac{f\left(x+h\right) - f\left(x\right)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(3x^2 + 3xh + h^2\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

So, $f'(x) = 3x^2$.

Definition (Alternate) – Derivative at a Point

-The derivative of a function f at a point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Applying the Alternate Definition

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

At the point $x = a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \bullet \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x \to a} \frac{x - a}{(x - a)\left(\sqrt{x} + \sqrt{a}\right)}$$

$$= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

-Applying this to an arbitrary $x > 0$ in the domain of f identifies the derivative as the function $f'(x) = \frac{1}{2\sqrt{x}}$ with domain $(0, \infty)$.

Notation

-There are many ways to denote the derivative of a function $y = f(x)$

y'	"y prime"
$\frac{dy}{dx}$	"dy dx" or "the derivative of y wrt x"
$\frac{df}{dx}$	"df dx" or "the derivative of f wrt x"
$\frac{d}{dx} f(x)$	"d dx of f at x" or "the derivative of f at x"